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A solution for hypersonic flow past slender bodies

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A solution to the hypersonic small disturbance equations is obtained for a class of two-dimensional bodies supporting logarithmic shock waves by reducing the partial differential equation to an ordinary differential equation. The body shape is calculated and shown not to be logarithmic if $\gamma \neq 1$, where γ is the ratio of the specific heats of the gas.

1. Introduction

It is well-known that under certain circumstances, partial differential equations may be reduced to ordinary differential equations, and the solutions so obtained are referred to as similarity solutions. Similarity solutions have been studied in many parts of gas dynamics including unsteady one-dimensional flow and steady hypersonic flow past slender bodies. In fact the equations of motion are the same in both cases. A detailed study has been made by Sedov (1959) for the case of power law shocks for which it is shown that similarity solutions exist when the pressure in the undisturbed stream is neglected. This was reviewed by Mirels (1962) with an emphasis on hypersonic flow over slender bodies. In Sedov's (1959) book, it is also indicated that a limiting case of similar flow fields with power law shock is the flow field formed with an exponential shock wave. For the case of two-dimensional hypersonic flow, such a special solution is studied by Cole & Aroesty (1970) using a different approach.

In this paper the equations describing hypersonic flow past a slender body are investigated and a single partial differential equation is derived. It is then shown that for a class of two-dimensional bodies supporting logarithmic shock waves this partial differential equation is reduced to an ordinary differential equation for which the solution is found. The body shape is calculated and shown not to be logarithmic if $\gamma \neq 1$. The solution is applied to hypersonic aerofoil calculation and the shape of the optimum aerofoils of this class is determined.

2. Formulation of the problem

To study hypersonic flow past two-dimensional or axisymmetric slender bodies, physical quantities are denoted with a bar, thus $(\bar{u}_\infty + \bar{u})$, $\bar{v}$ are velocity components in the $\bar{x}$ and $\bar{r}$ directions (for two-dimensional flow, $\bar{r}$ is simply $\bar{y}$), $\bar{p}$ and $\bar{\rho}$, the pressure and density, respectively.

Let ϵ be a measure of the slenderness of the body. Non-dimensional quantities are introduced as follows

$$x = \bar{x}, \quad r = \bar{r}/\epsilon, \tag{1}$$

$$\left.\begin{aligned} u &= \bar{u}/\bar{u}_\infty \epsilon^2, & v &= \bar{v}/\bar{u}_\infty \epsilon, \\ p &= \bar{p}/\gamma \bar{p}_\infty M_\infty^2 \epsilon^2, & \rho &= \bar{\rho}/\bar{\rho}_\infty, \end{aligned}\right\} \tag{2}$$

where γ is the ratio of specific heats, M the Mach number and the subscript ∞ refers to free stream. These non-dimensional quantities are assumed to be of order unity in the flow region which is of interest and when terms of order ϵ^2 are neglected, we obtain the following hypersonic small disturbance equations†

$$\rho_x + (\rho v)_r + \sigma \rho v / r = 0, \tag{3}$$

$$v_x + v v_r + p_r/\rho = 0, \tag{4}$$

$$(p/\rho^\gamma)_x + v(p/\rho^\gamma)_r = 0, \tag{5}$$

with $\sigma = 0$ for two-dimensional flow and $\sigma = 1$ for axisymmetric flow.

Introducing a stream function ψ such that

$$\psi_r = \rho r^\sigma, \quad \psi_x = -\rho v r^\sigma, \tag{6}$$

equation (3) is automatically satisfied while equation (5) becomes

$$p = \omega(\psi)\rho^\gamma. \tag{7}$$

Using (6) and (7), equation (4) can be written as‡

$$\psi_r^2 \psi_{xx} - 2\psi_r \psi_x \psi_{rx} + \left[\psi_x^2 - \frac{\gamma\omega}{r^{\sigma(\gamma-1)}}\psi_r^{\gamma+1}\right]\psi_{rr} + \frac{\psi_r^{\gamma+2}}{r^{\sigma\gamma}}(\sigma\gamma\omega - r^\sigma \omega' \psi_r) = 0. \tag{8}$$

When r is considered as a function of x and ψ, the last equation becomes

$$\frac{r_{xx}}{r^\sigma} + \left[\frac{\omega}{(r^\sigma r_\psi)^\gamma}\right]_\psi = 0. \tag{9}$$

The derivatives of r are related to the flow quantities by

$$r_x = v, \quad r_\psi = 1/\rho r^\sigma. \tag{10}$$

The direct problem of finding the flow field and the bow shock, from (9), for a given body is difficult and we shall treat only the indirect problem, i.e. for a given bow shock we seek, from (9), a flow field and a body which supports the given shock. In this case the function $\omega(\psi)$ can be written explicitly in terms of the shock. Thus if the shock is given by

$$r = R(x) \tag{11}$$

and if the pressure in the free stream is neglected, we have, from the Rankine–Hugoniot conditions,

$$\omega = \omega_0 \Omega^2, \tag{12}$$

where

$$\left.\begin{aligned} \omega_0 &= \frac{2}{\gamma+1}\left(\frac{\gamma-1}{\gamma+1}\right)^\gamma, \\ \Omega &= R'[R^{-1}\{((1+\sigma)\psi)^{1/(1+\sigma)}\}]. \end{aligned}\right\} \tag{13}$$

In (13), R^{-1} denotes the inverse function of R.

† Subscripts x and r represent partial differentiation.

‡ A prime denotes total derivative.

The structure of the function $\omega(\psi)$ suggests the following transformation

$$\psi = \frac{1}{1+\sigma}[R(\xi)]^{1+\sigma}. \tag{14}$$

The new variable ξ defined in (14) is just the x location of a streamline where it crosses the shock (see figure 1), thus at the shock, $\xi = x$, and for the region behind the shock, $\xi < x$. For the special case $\sigma = 0$, this transformation was used by Cole & Aroesty (1970) in a recent paper.

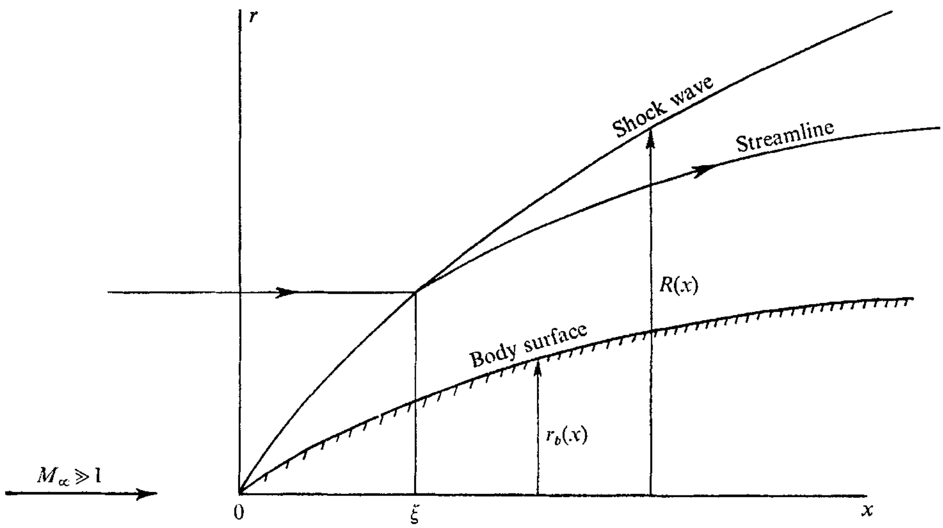

FIGURE 1. System of co-ordinates.

Transformation (14) simplifies the expression for ω and we have

$$\omega = \omega_0 R'^2(\xi), \tag{15}$$

hence equation (9) becomes

$$\frac{r_{xx}}{r^\sigma} + \frac{\omega_0}{R^\sigma(\xi)\,R'(\xi)}\frac{\partial}{\partial\xi}\left[\frac{R'^2(\xi)}{\{r^\sigma r_\xi/R^\sigma(\xi)\,R'(\xi)\}^\gamma}\right] = 0, \tag{16}$$

for which two boundary values at the shock, i.e. $r(x,x)$ and either $r_\xi(x,x)$ or $r_x(x,x)$, must be prescribed in order to determine a unique solution.

In passing we notice that

$$r^\sigma r_\xi/R^\sigma(\xi)\,R'(\xi) = 1/\rho \tag{17}$$

and this expression must always be positive.

It can be shown that solutions of (16) exist in the form $r = R(x)f[\eta(\xi,x)]$. Indeed, this leads to the well-known power law similarity solutions and its limiting case – flow with an exponential bow shock. These will not be repeated here.

3. Method of solution

In this section we shall restrict our discussion to the case $\sigma = 0$, i.e. two-dimensional hypersonic flow past a slender body. Solutions to (16) will be sought in the form

$$r = R(x) - f[\eta(\xi,x)] \tag{18}$$

and it will be shown that (16) is reduced to an ordinary differential equation for $f(\eta)$, provided certain conditions are satisfied.

First, the conditions imposed on η and f are that η must be a constant along the shock, i.e.

$$\eta(x, x) = 1 \tag{19}$$

and

$$f(1) = 0. \tag{20}$$

For the case $\sigma = 0$, (16) becomes

$$r_{xx} + \frac{\omega_0}{R'(\xi)} \frac{\partial}{\partial \xi}\left[\frac{R'^2(\xi)}{\{r_\xi / R'(\xi)\}^\gamma}\right] = 0. \tag{21}$$

Using (18) we have

$$\frac{r_\xi}{R'(\xi)} = -\frac{\eta_\xi}{R'(\xi)} f'.$$

For this expression to be a function of η only, a condition on η is

$$\eta_\xi = -R'(\xi)\, G(\eta) \tag{22}$$

and thus

$$r_\xi / R'(\xi) = f'G. \tag{23}$$

Equation (21) then becomes

$$\frac{R''(x)}{R'^2(\xi)} - \left(\frac{\eta_x}{R'(\xi)}\right)^2 f'' - \frac{\eta_{xx}}{R'^2(\xi)} f' + \frac{R''(\xi)}{R'^2(\xi)} \frac{2\omega_0}{(Gf')^\gamma} + \frac{\gamma\omega_0 G(Gf'' + f'G')}{(Gf')^{\gamma+1}} = 0. \tag{24}$$

The conditions required for (24) to become an ordinary differential equation are thus

$$R''(\xi)/R'^2(\xi) = \text{const.} \equiv -\beta, \tag{25}$$

$$R''(x)/R'^2(\xi) = \text{a function of } \eta \text{ only} \tag{26}$$

and

$$\eta_x = R'(\xi)\, F(\eta). \tag{27}$$

The special case $\beta = 0$ corresponds to flow past a wedge and need not be discussed here. For $\beta \neq 0$ the solution of (25) is

$$R(\xi) = \frac{1}{\beta} \ln\left(1 + \frac{\xi}{A}\right), \tag{28}$$

where A is a constant and $A\beta > 0$. In deriving (28) the condition that the shock passes through the origin, i.e. $R(0) = 0$, has been used. Thus the shock wave must be logarithmic.

With this shock wave given, the conditions (22), (26) and (27) will all be satisfied for

$$\eta = \frac{x + A}{\xi + A}, \quad F = \beta, \quad G = \beta\eta \tag{29}$$

and, after introducing

$$h(\eta) = \beta f'(\eta), \tag{30}$$

(24) becomes a first-order equation in h

$$\frac{dh}{d\eta} = \frac{h[(2-\gamma)\,\omega_0 \eta^{2-\gamma} + h^\gamma]}{\gamma\omega_0 \eta^{3-\gamma} - \eta^2 h^{\gamma+1}}, \tag{31}$$

with the boundary condition at shock

$$h(1) = \frac{\gamma - 1}{\gamma + 1}, \tag{32}$$

which is obtained by combining (23), (29), (30) and (17).

After $h(\eta)$ is found from (31), the flow quantities can be obtained from

$$\left.\begin{aligned} \rho &= 1/\eta h(\eta), \\ v &= R'(x)(1 - 1/\rho), \\ p &= \omega_0 R'^2(\xi)\rho^\gamma, \end{aligned}\right\} \tag{33}$$

and in particular, the body surface, given by $\xi = 0$, is

$$r = r_b(x) = \frac{1}{\beta}\left[\ln\left(1 + \frac{x}{A}\right) - \int_1^{1+(x/A)} h(\eta)\, d\eta\right]. \tag{34}$$

It is easily seen from (34) that for $\gamma \neq 1$ the body which supports a logarithmic bow shock is itself not logarithmic,† since this would require that $h \sim 1/\eta$ which is nevertheless not a solution of (31) when $\gamma \neq 1$. The case $\gamma = 1$ will be discussed later.

Scaling the body surface by the normalization condition

$$r_b(1) = 1 \tag{35}$$

relates the two parameters β and A by

$$\beta = \ln\left(1 + \frac{1}{A}\right) - \int_1^{1+(1/A)} h(\eta)\, d\eta. \tag{36}$$

Hence the profile of the body surface is a family of curves of one parameter. In the limiting case $|A| \to \infty$ we have

$$A\beta \to 2/(\gamma + 1), \quad \text{as } |A| \to \infty, \tag{37}$$

and the body surface becomes a flat plate as seen from (34).

It is clear that everything depends on the solution of (31). For the limiting case $\gamma \to 1$, the solution to (31) which satisfies the boundary condition (32) is

$$h = 0. \tag{38}$$

This, according to (33) and (34), gives an infinite density and a logarithmic body surface coincident with the shock. These conclusions are of course all consistent with the concept of Newtonian flow.

In the general case $\gamma \neq 1$, and (31) must be solved numerically. It should be noted that h is independent of the parameter β or A, but dependent only on γ. Hence h is a universal function for all bodies and can be computed once γ is given.

For concave shocks and thus concave body shapes, $\beta < 0$, $A < 0$ and $\eta < 1$ in the region behind the shock. Evidently, for a given value of $A < 0$, these solutions exist only in the region

$$0 < x < -A. \tag{39}$$

† As the body and the shock wave are not similar in shape, this particular type of solution is not referred to as a similar solution.

As $x \to -A$, the slopes of the shock and the body surface, as seen from (28) and (34), tend to infinity, and the basic assumptions in the hypersonic small disturbance theory break down. Therefore for concave shapes, (31) must be solved in the range

$$0 < \eta < 1. \tag{40}$$

As $\eta \to 0$, the asymptotic behaviour of h is obtained from (31) as follows

$$h = \eta^{(2/\gamma)-1}\left(\text{const.} - \frac{1}{\omega_0}\ln\eta\right)^{-1/\gamma}, \tag{41}$$

which may be used in the numerical solution of (31).

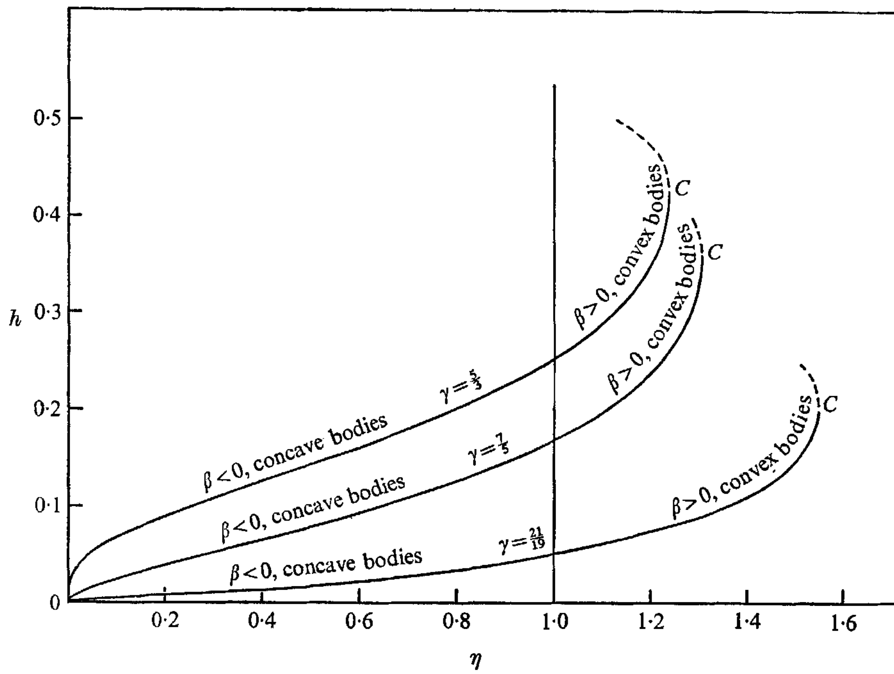

FIGURE 2. The universal function $h(\eta)$ (see equation (31)).

Since for a given value of $A < 0$, the solution so obtained only exists over the body surface from $x = 0$ up to $x = -A$, if $-A < 1$ the normalization condition (35) must be replaced by a slightly different one, e.g. $r_b(-\frac{1}{2}A) = 1$.

For convex shocks and thus convex body shapes, the situation is different. In this case, $\beta > 0$, $A > 0$ and $\eta > 1$ in the region behind the shock. Equation (31) then shows clearly that the integral curve (figure 2) can only be continued from $\eta = 1$ to $\eta = \eta_c$ where η_c is the η co-ordinate of the point C where the integral curve meets the curve Γ defined by

$$\Gamma\colon\ \eta^{\gamma-1}h^{\gamma+1} - \gamma\omega_0 = 0. \tag{42}$$

The integral curve turns around at the point C. Thus the line $\eta = \eta_c$ represents a limit line of the flow field, and that part of the integral curve (shown by a broken line in figure 2) above the limit point C must be rejected on physical ground.

The local behaviour of the integral curve at point $C(\eta_c, h_c)$ can be investigated using (31). Thus putting

$$\eta^* = \eta - \eta_c, \quad h^* = h - h_c, \tag{43}$$

(31) becomes, for small values of η^* and h^*,

$$d\eta^*/dh^* = -a\eta^* - bh^*, \tag{44}$$

where

$$\left.\begin{aligned} a &= \frac{(\gamma-1)\,\eta_c h_c^\gamma}{(2-\gamma)\,\omega_0 \eta_c^{2-\gamma} + h_c^\gamma}, \\ b &= \frac{(\gamma+1)\,\eta_c^2 h_c^{\gamma-1}}{(2-\gamma)\,\omega_0 \eta_c^{2-\gamma} + h_c^\gamma}, \end{aligned}\right\} \tag{45}$$

and the local behaviour of the integral curve near C is found as follows

$$\eta - \eta_c = -\tfrac{1}{2}b(h-h_c)^2, \tag{46}$$

which shows clearly that it turns around at the limit point C. As $\eta \to \eta_c$, $dh/d\eta \to \infty$ and hence, from (33), $(\partial v/\partial x)_{\xi=0} \to \infty$, therefore the curvature of the body is infinite at C.

From these discussions, it is clear that, in the case of convex shapes, for the solution to exist up to the point $x = 1$, the condition on A is

$$A > 1/(\eta_c - 1) \tag{47}$$

and if A is less than $1/(\eta_c - 1)$, the solution cannot exist up to $x = 1$. However, the solution so obtained still makes sense and represents flow over the front part of the body up to the limiting point $x = A(\eta_c - 1)$ with a slightly different normalization condition from (35).

The universal function $h(\eta)$ has been computed from (31) in both cases of concave shapes and convex shapes. The results are shown in figure 2 for various values of γ. Some typical body shapes supporting logarithmic shock waves are given in figures 3 and 4.

4. Hypersonic airfoils supporting logarithmic shock waves

The shape of body surface found in § 3 can be used as the lower surface of a hypersonic airfoil at small incidence. For such an airfoil the pressure distribution $p_b(x)$ on the lower surface is given by

$$p_b(x) = \frac{\omega_0}{(A\beta)^2}\left[1\bigg/\left(1+\frac{x}{A}\right) h\left(1+\frac{x}{A}\right)\right]^\gamma \tag{48}$$

while that on the upper surface is assumed zero. Within the assumption of an inviscid fluid, the lift L and drag D, and the coefficients C_L and C_D, of the airfoil can be obtained as follows

$$\left.\begin{aligned} C_L &\equiv \frac{L}{\frac{1}{2}\bar{\rho}_\infty \bar{u}_\infty^2} = \epsilon^2 \frac{2\omega_0}{A\beta^2}\int_1^{1+(1/A)} \left(\frac{1}{\eta h}\right)^\gamma d\eta, \\ C_D &= \frac{D}{\frac{1}{2}\bar{\rho}_\infty \bar{u}_\infty^2} = \epsilon^3 \frac{2\omega_0}{A^2\beta^3}\int_1^{1+(1/A)} \left(\frac{1}{\eta h}\right)^\gamma \left(\frac{1}{\eta} - h\right) d\eta. \end{aligned}\right\} \tag{49}$$

Evidently $C_L^{\frac{3}{2}}/C_D$ is independent of ϵ and is a function of the parameter A. The problem of determining the shape of an airfoil which produces a minimum

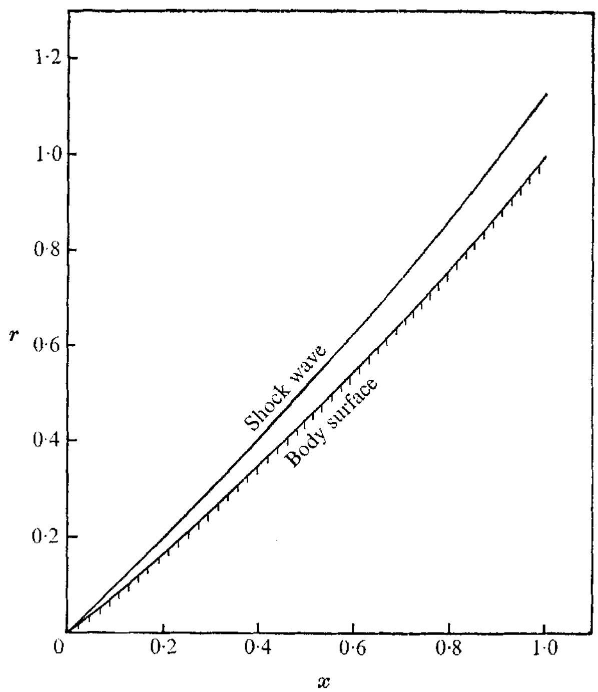

FIGURE 3. Optimum shape of airfoil supporting a logarithmic shock $\gamma = 1{\cdot}4$, $A = -3{\cdot}33$, $C_L^{\frac{3}{2}}/C_D = 1{\cdot}58$.

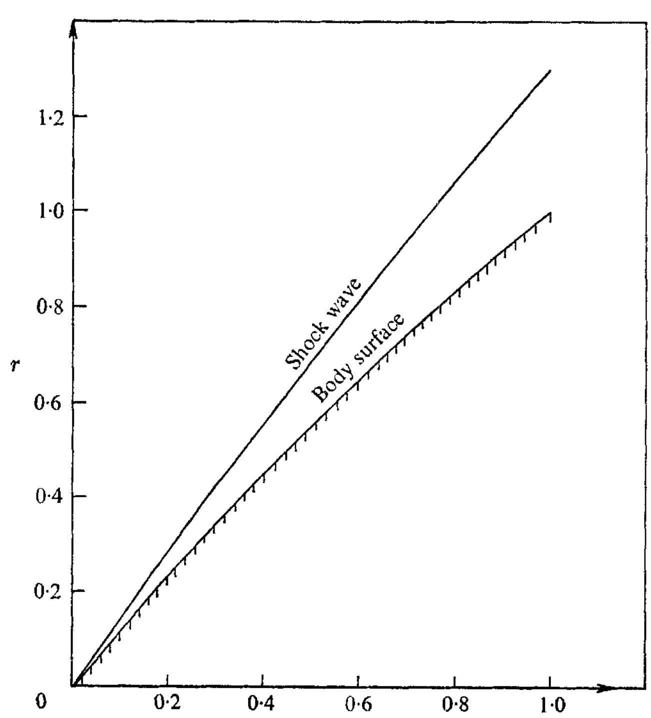

FIGURE 4. A typical convex hypersonic airfoil supporting a logarithmic shock $\gamma = 1{\cdot}4$, $A = 4{\cdot}0$, $C_L^{\frac{3}{2}}/C_D = 1{\cdot}48$.

drag for a given lift can thus be studied in terms of this parameter by maximizing the quantity $C_L^{\frac{3}{2}}/C_D$. In figure 5 the curves of $C_L^{\frac{3}{2}}/C_D$ *versus* $1/A$ are plotted for

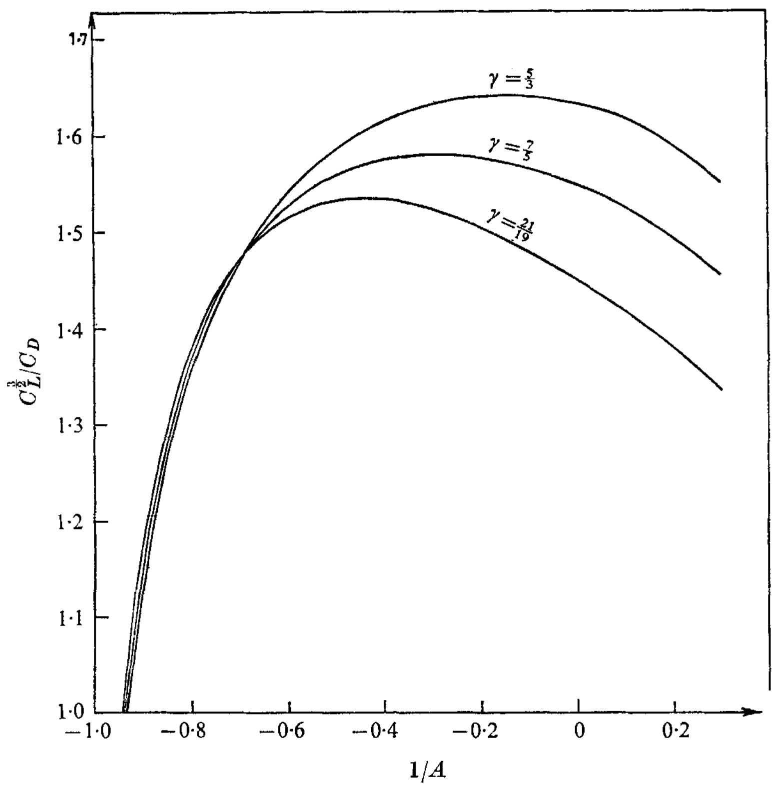

FIGURE 5. C_L/C_D as a function of $1/A$.

various values of γ. It is seen from this figure that for a given lift an airfoil of this type has slightly lower drag than a flat plate, for which

$$1/A = 0 \quad \text{and} \quad C_L^{\frac{3}{2}}/C_D = (\gamma+1)^{\frac{1}{2}}.$$

It is also seen that with decreasing γ, the improvement in performance of such an airfoil over a flat plate is increasing. For $\gamma = 1{\cdot}4$, the optimum shape of airfoils of this type which gives a maximum $C_L^{\frac{3}{2}}/C_D$ is plotted in figure 3.

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